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Physical States on a C^*-algebra

(Abstract)

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Let A be a C^*- algebra. A <u>physical state</u> ρ on A is a function $\rho: A \to \mathbb{C}$ such that ρ restricted to any C^*- subalgebra B generated by a single self-adjoint element, is linear and positive. Moreover we require

$$1 = \sup \{\rho(a) : a \in A ; a \geq 0 ; \|a\| \leq 1\}$$

A physical state need not be linear on A. Indeed, assume A to be non-abelian (a similar example works in the abelian case), and let x be a non-normal element of A. Lett ϕ be a positive linear functional of norm one on A, such that $\phi(x^*x - xx^*) \neq 0$. Define, for $a \in A$.

$$\rho(a) = \phi(a) + \phi(a^*a - aa^*)$$

Then ρ coincides with ϕ on any abelian C^*- subalgebra of A, so in particular ρ is a physical state on A, but ρ is not linear.

A <u>quasi-state</u> is a physical state ρ which also satisfies

$$\rho(a) = \rho(a_1) + i\rho(a_2)$$

for all $a \in A$, where $a = a_1 + ia_2$ is the canonical decomposition of a in self-adjoint parts. The problem discussed is under what conditions a quasi-state is linear, i.e. when it is a state in the traditional sense. One might be tempted to believe that this is always true, but it is possible to give an example showing it to be false for the 2×2 matrix algebra.

In somewhat less general form, this problem was first proposed by Mackey (2), and solved in a very special case by Gleason (1).

In its present form, the problem is due to R.V. Kadison. It has bearings on quantum mechanics, where it corresponds to the question of linearity of the expectation-functional on the algebra of observables. It is also related to the question of additivity of the trace on a II_1-factor.

One may show that a quasi-state on an abelian C^*-algebra is always linear. If H is a Hilbert-space of dimension ≥ 3, the same is true for quasi-states on $\mathcal{LC}$ (H) = the compact operators on H, and for quasi-states on any C^*-subalgebra A of $\mathcal{LC}$ (H), provided A has no two-dimensional irreducible representations.

As one does for states, one may introduce <u>pure</u> quasi-states. It can be shown that each pure and norm-continuous quasi-state a CCR-algebra A is linear if Prim A is Hausdorff. Some other special cases are also within reach, but a general solution seems to be at a safe distance.

(1) A.M. Gleason, Measures on the closed subspaces of a Hilbert-space. Jour.Math. and Mech. 6(6)(1957) 885-893.

(2) G.W. Mackey, Mathematical Foundations of quantum mechanics. W.A. Benjamin Inc. New York, N.Y. (1963).